JACOBI-STIRLING POLYNOMIALS AND P-PARTITIONS

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ABSTRACT. We investigate the diagonal generating function of the Jacobi-Stirling numbers of the second kind $\mathrm{JS}(n+k,n;z)$ by generalizing the analogous results for the Stirling and Legendre-Stirling numbers. More precisely, letting $\mathrm{JS}(n+k,n;z)=p_{k,0}(n)+p_{k,1}(n)z+\cdots+p_{k,k}(n)z^k$, we show that $(1-t)^{3k-i+1}\sum_{n\geq 0}p_{k,i}(n)t^n$ is a polynomial in t with nonnegative integral coefficients and provide combinatorial interpretations of the coefficients by using Stanley's theory of P-partitions.

1. Introduction

Let $\ell_{\alpha,\beta}[y](t)$ be the Jacobi differential operator:

$$\ell_{\alpha,\beta}[y](t)=\frac{1}{(1-t)^\alpha(1+t)^\beta}\left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t)\right)'.$$

It is well known that the Jacobi polynomial $y=P_n^{(\alpha,\beta)}(t)$ is an eigenvector for the differential operator $\ell_{\alpha,\beta}$ corresponding to $n(n+\alpha+\beta+1)$, i.e.,

$$\ell_{\alpha,\beta}[y](t)=n(n+\alpha+\beta+1)y(t).$$

For each $n\in\mathbb{N}$, the Jacobi-Stirling numbers $\mathrm{JS}(n,k;z)$ of the second kind appeared originally as the coefficients in the expansion of the n-th composite power of $\ell_{\alpha,\beta}$ (see [7]):

$$(1-t)^\alpha(1+t)^\beta\ell^n_{\alpha,\beta}[y](t)=\sum_{k=0}^{n}(-1)^k\,\mathrm{JS}(n,k;z)\left((1-t)^{\alpha+k}(1+t)^{\beta+k}y^{(k)}(t)\right)^{(k)},$$

where $z=\alpha+\beta+1$, and can also be defined as the connection coefficients in

$$x^n=\sum_{k=0}^{n}\mathrm{JS}(n,k;z)\prod_{i=0}^{k-1}(x-i(z+i)). \tag{1.1}$$

The Jacobi-Stirling numbers $\mathrm{js}(n,k;z)$ of the first kind are defined by

$$\prod_{i=0}^{n-1}(x-i(z+i))=\sum_{k=0}^{n}\mathrm{js}(n,k;z)x^k. \tag{1.2}$$

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When $z = 1$, the Jacobi-Stirling numbers become the *Legendre-Stirling numbers* [6] of the first and second kinds:

$$\mathrm{ls}(n,k) = \mathrm{js}(n,k;1), \quad \mathrm{LS}(n,k) = \mathrm{JS}(n,k;1). \tag{1.3}$$

Generalizing the work of Andrews and Littlejohn [2] on Legendre-Stirling numbers, Gelineau and Zeng [9] studied the combinatorial interpretations of the Jacobi-Stirling numbers and remarked on the connection with Stirling numbers and central factorial numbers. Further properties of the Jacobi-Stirling numbers have been given by Andrews, Egge, Gawronski, and Littlejohn [1].

The Stirling numbers of the second and first kinds $S(n,k)$ and $s(n,k)$ are defined by

$$x^n = \sum_{k=0}^{n} S(n,k) \prod_{i=0}^{k-1}(x-i), \qquad \prod_{i=0}^{n-1}(x-i) = \sum_{k=0}^{n} s(n,k)x^k. \tag{1.4}$$

The lesser known central factorial numbers [14, p. 213–217] $T(n,k)$ and $t(n,k)$ are defined by

$$x^n = \sum_{k=0}^{n} T(n,k)\, x \prod_{i=1}^{k-1}\left(x + \frac{k}{2} - i\right), \tag{1.5}$$

and

$$x \prod_{i=1}^{n-1}\left(x + \frac{n}{2} - i\right) = \sum_{k=0}^{n} t(n,k)x^k. \tag{1.6}$$

Starting from the fact that for fixed k, the Stirling number $S(n+k,n)$ can be written as a polynomial in n of degree $2k$ and there exist nonnegative integers $c_{k,j}$, $1 \le j \le k$, such that

$$\sum_{n\ge 0} S(n+k,n)t^n = \frac{\sum_{j=1}^{k} c_{k,j}t^j}{(1-t)^{2k+1}}, \tag{1.7}$$

Gessel and Stanley [10] gave a combinatorial interpretation for the $c_{k,j}$ in terms of the descents in *Stirling permutations*. Recently, Egge [5] has given an analogous result for the Legendre-Stirling numbers, and Gelineau [8] has made a preliminary study of the analogous problem for Jacobi-Stirling numbers. In this paper, we will prove some analogous results for the diagonal generating function of Jacobi-Stirling numbers. As noticed in [9], the leading coefficient of the polynomial $\mathrm{JS}(n,k;z)$ is $S(n,k)$ and the constant term of $\mathrm{JS}(n,k;z)$ is the central factorial number of the second kind with even indices $T(2n,2k)$. Similarly, the leading coefficient of the polynomial $\mathrm{js}(n,k;z)$ is $s(n,k)$ and the constant term of $\mathrm{js}(n,k;z)$ is the central factorial number of the first kind with even indices $t(2n,2k)$.

Definition 1. *The* Jacobi-Stirling polynomial *of the second kind is defined by*

$$f_k(n;z) := \mathrm{JS}(n+k,n;z). \tag{1.8}$$

The coefficient $p_{k,i}(n)$ *of* z^i *in* $f_k(n;z)$ *is called the* Jacobi-Stirling coefficient *of the second kind for* $0 \le i \le k$. *Thus*

$$f_k(n;z) = p_{k,0}(n) + p_{k,1}(n)z + \cdots + p_{k,k}(n)z^k. \tag{1.9}$$

The main goal of this paper is to prove Theorems 1 and 2 below.

Theorem 1. *For each integer* k *and* i *such that* $0 \le i \le k$, *there is a polynomial* $A_{k,i}(t) = \sum_{j=1}^{2k-i} a_{k,i,j}t^j$ *with positive integer coefficients such that*

$$\sum_{n\ge 0} p_{k,i}(n)t^n = \frac{A_{k,i}(t)}{(1-t)^{3k-i+1}}. \tag{1.10}$$

In order to give a combinatorial interpretation for $a_{k,i,j}$, we introduce the multiset

$$M_k := \{1,1,\bar{1},2,2,\bar{2},\ldots,k,k,\bar{k}\},$$

where the elements are ordered by

$$\bar{1} < 1 < \bar{2} < 2 \ldots < \bar{k} < k. \tag{1.11}$$

Let $[\bar{k}] := \{\bar{1},\bar{2},\ldots,\bar{k}\}$. For any subset $S \subseteq [\bar{k}]$, we set $M_{k,S} = M_k \setminus S$.

Definition 2. *A permutation* π *of* $M_{k,S}$ *is a Jacobi-Stirling permutation if whenever* $u < v < w$ *and* $\pi(u) = \pi(w)$, *we have* $\pi(v) > \pi(u)$. *We denote by* $\mathcal{JSP}_{k,S}$ *the set of Jacobi-Stirling permutations of* $M_{k,S}$ *and*

$$\mathcal{JSP}_{k,i} = \bigcup_{\substack{S\subseteq[\bar{k}]\\ |S|=i}} \mathcal{JSP}_{k,S}.$$

For example, the Jacobi-Stirling permutations of $\mathcal{JSP}_{2,1}$ are:

$$22\bar{2}11,\ \bar{2}2211,\ \bar{2}1221,\ \bar{2}1122,\ 221\bar{2}1,\ 122\bar{2}1,\ 1\bar{2}221, 1\bar{2}122,\ 2211\bar{2},\ 1221\bar{2},$$
$$1122\bar{2},\ 11\bar{2}22,\ 2211\bar{1},\ 1221\bar{1},\ 1122\bar{1},\ 11\bar{1}22,\ 22\bar{1}11,\ \bar{1}2211,\ \bar{1}1221,\ \bar{1}1122.$$

Let $\pi = \pi_1\pi_2\ldots\pi_m$ be a word on a totally ordered alphabet. We say that π has a descent at l, where $1 \le l \le m-1$, if $\pi_l > \pi_{l+1}$. Let des π be the number of descents of π. The following is our main interpretation for the coefficients $a_{k,i,j}$.

Theorem 2. *For* $k \ge 1$, $0 \le i \le k$, *and* $1 \le j \le 2k-i$, *the coefficient* $a_{k,i,j}$ *is the number of Jacobi-Stirling permutations in* $\mathcal{JSP}_{k,i}$ *with* $j-1$ *descents.*

The rest of this paper is organized as follows. In Section 2, we investigate some elementary properties of the Jacobi-Stirling polynomials and prove Theorem 1. In Section 3 we apply Stanley's P-partition theory to derive a first interpretation of the integers $a_{k,i,j}$ and then reformulate it in terms of descents of Jacobi-Stirling permutations in Section 4.

In Section 5, we construct Legendre-Stirling posets in order to prove a similar result for the Legendre-Stirling numbers, and then to deduce Egge's result for Legendre-Stirling numbers [5] in terms of descents of Legendre-Stirling permutations. A second proof of Egge's result is given by making a link to our result for Jacobi-Stirling permutations, namely Theorem 2. We end this paper with a conjecture on the real-rootedness of the polynomials $A_{k,i}(t)$.

2. Jacobi-Stirling polynomials

Proposition 3. *For $0 \le i \le k$, the Jacobi-Stirling coefficient $p_{k,i}(n)$ is a polynomial in n of degree $3k-i$. Moreover, the leading coefficient of $p_{k,i}(n)$ is*

$$\frac{1}{3^{k-i}2^i\, i!\,(k-i)!} \tag{2.1}$$

for all $0 \le i \le k$.

Proof. We proceed by induction on $k \ge 0$. For $k=0$, we have $p_{0,0}(n)=1$ since $f_0(n) = \mathrm{JS}(n,n;z)=1$. Let $k\ge 1$ and suppose that $p_{k-1,i}$ is a polynomial in n of degree $3(k-1)-i$ for $0 \le i \le k-1$. From (1.1) we deduce the recurrence relation:

$$\begin{cases} \mathrm{JS}(0,0;z)=1, \qquad \mathrm{JS}(n,k;z)=0, \text{ if } k \notin \{1,\dots,n\}, \\ \mathrm{JS}(n,k;z)=\mathrm{JS}(n-1,k-1;z)+k(k+z)\ \mathrm{JS}(n-1,k;z), \text{ for } n,k\ge 1. \end{cases} \tag{2.2}$$

Substituting in (1.8) yields

$$f_k(n;z)-f_k(n-1;z)=n(n+z)f_{k-1}(n;z). \tag{2.3}$$

It follows from (1.9) that for $0\le i\le k$,

$$p_{k,i}(n)-p_{k,i}(n-1)=n^2p_{k-1,i}(n)+np_{k-1,i-1}(n). \tag{2.4}$$

Applying the induction hypothesis, we see that $p_{k,i}(n)-p_{k,i}(n-1)$ is a polynomial in n of degree at most

$$\max(3(k-1)-i+2, 3(k-1)-(i-1)+1) = 3k-i-1.$$

Hence $p_{k,i}(n)$ is a polynomial in n of degree at most $3k-i$. It remains to determine the coefficient of n^{3k-i}, say $\beta_{k,i}$. Extracting the coefficient of n^{3k-i-1} in (2.4) we have

$$\beta_{k,i}=\frac{1}{3k-i}(\beta_{k-1,i}+\beta_{k-1,i-1}).$$

Now it is fairly easy to see that (2.1) satisfies the above recurrence. □

Proposition 4. *For all $k\ge 1$ and $0\le i\le k$, we have*

$$p_{k,i}(0)=p_{k,i}(-1)=p_{k,i}(-2)=\cdots=p_{k,i}(-k)=0. \tag{2.5}$$

TABLE 1. The first values of $A_{k,i}(t)$

$k\backslash i$	0	1	2	3
0	1			
1	$t+t^2$	t		
2	$t+14t^2+21t^3+4t^4$	$2t+12t^2+6t^3$	$t+2t^2$	
3	$t+75t^2+603t^3+1065t^4+460t^5+36t^6$	$3t+114t^2+501t^3+436t^4+66t^5$	$3t+55t^2+116t^3+36t^4$	$t+8t^2+6t^3$

Proof. We proceed by induction on k. By definition, we have

$$f_1(n;z) = \mathrm{JS}(n+1,n;z) = p_{1,0}(n) + p_{1,1}(n)z.$$

As noticed in [9, Theorem 1], the leading coefficient of the polynomial $\mathrm{JS}(n,k;z)$ is $S(n,k)$ and the constant term is $T(2n,2k)$. We derive from (1.4) and (1.5) that

$$p_{1,1}(n) = S(n+1,n) = n(n+1)/2,$$
$$p_{1,0}(n) = T(2n+2,2n) = n(n+1)(2n+1)/6.$$

Hence (2.5) is true for $k=1$. Assume that (2.5) is true for some $k \geq 1$. By (2.4) we have

$$p_{k,i}(n) - p_{k,i}(n-1) = n^2 p_{k-1,i}(n) + n p_{k-1,i-1}(n).$$

Since $\mathrm{JS}(0,k;z)=0$ if $k \geq 2$, we have $p_{k,i}(0)=0$. The above equation and the induction hypothesis imply successively that

$$p_{k,i}(-1)=0, \quad p_{k,i}(-2)=0, \ \ldots, \ p_{k,i}(-k+1)=0, \quad p_{k,i}(-k)=0.$$

The proof is thus complete. □

Lemma 5. *For each integer k and i such that $0 \leq i \leq k$, there is a polynomial $A_{k,i}(t) = \sum_{j=1}^{2k-i} a_{k,i,j} t^j$ with integer coefficients such that*

$$\sum_{n\geq 0} p_{k,i}(n)t^n = \frac{A_{k,i}(t)}{(1-t)^{3k-i+1}}. \tag{2.6}$$

Proof. By Proposition 3 and standard results concerning rational generating functions (cf. [16, Corollary 4.3.1]), for each integer k and i such that $0 \leq i \leq k$, there is a polynomial $A_{k,i}(t) = a_{k,i,0} + a_{k,i,1}t + \cdots + a_{k,i,3k-i}t^{3k-i}$ satisfying (2.6). Now, by [16, Proposition 4.2.3], we have

$$\sum_{n\geq 1} p_{k,i}(-n)t^n = -\frac{A_{k,i}(1/t)}{(1-1/t)^{2k-i+1}}. \tag{2.7}$$

Applying (2.5) we see that $a_{k,i,2k-i+1} = \cdots = a_{k,i,3k-i} = 0$. □

The first values of $A_{k,i}(t)$ are given in Table 1. The following result gives a recurrence for the coefficients $a_{k,i,j}$.

Proposition 6. *Let $a_{0,0,0} = 1$. For $k, i, j \geq 0$, we have the following recurrence for the integers $a_{k,i,j}$:*

$$\begin{aligned} a_{k,i,j} = j^2 a_{k-1,i,j} + [2(j-1)(3k-i-j-1) + (3k-i-2)]a_{k-1,i,j-1} \\ +(3k-i-j)^2 a_{k-1,i,j-2} + j a_{k-1,i-1,j} + (3k-i-j)a_{k-1,i-1,j-1}, \end{aligned} \tag{2.8}$$

where $a_{k,i,j} = 0$ if any of the indices k, i, j is negative or if $j \notin \{1, \ldots, 2k-i\}$.

Proof. For $0 \leq i \leq k$, let

$$F_{k,i}(t) = \sum_{n \geq 0} p_{k,i}(n) t^n = \frac{A_{k,i}(t)}{(1-t)^{3k-i+1}}. \tag{2.9}$$

The recurrence relation (2.4) is equivalent to

$$F_{k,i}(t) = (1-t)^{-1}[t^2 F''_{k-1,i}(t) + tF'_{k-1,i}(t) + tF'_{k-1,i-1}(t)] \tag{2.10}$$

with $F_{0,0} = (1-t)^{-1}$. Substituting (2.9) into (2.10) we obtain

$$\begin{aligned} A_{k,i}(t) =& (1-t)^{3k-i}[t^2(A_{k-1,i}(t)(1-t)^{-(3k-i-2)})'' \\ &+ t(A_{k-1,i}(t)(1-t)^{-(3k-i-2)})' + t(A_{k-1,i-1}(t)(1-t)^{-(3k-i-1)})'] \\ =& [t^2 A''_{k-1,i}(t)(1-t)^2 + 2(3k-i-2)t^2 A'_{k-1,i}(t)(1-t) \\ &+ (3k-i-2)(3k-i-1)t^2 A_{k-1,i}(t)] \\ &+ [tA'_{k-1,i}(t)(1-t)^2 + (3k-i-2)tA_{k-1,i}(t)(1-t)] \\ &+ [tA'_{k-1,i-1}(t)(1-t) + (3k-i-1)tA_{k-1,i-1}(t)]. \end{aligned}$$

Taking the coefficient of t^j in both sides of the above equation, we have

$$\begin{aligned} a_{k,i,j} =& j(j-1)a_{k-1,i,j} - 2(j-1)(j-2)a_{k-1,i,j-1} + (j-2)(j-3)a_{k-1,i,j-2} \\ &+ 2(3k-i-2)(j-1)a_{k-1,i,j-1} - 2(3k-i-2)(j-2)a_{k-1,i,j-2} \\ &+ (3k-i-2)(3k-i-1)a_{k-1,i,j-2} + ja_{k-1,i,j} - 2(j-1)a_{k-1,i,j-1} \\ &+ (j-2)a_{k-1,i,j-2} + (3k-i-2)a_{k-1,i,j-1} - (3k-i-2)a_{k-1,i,j-2} \\ &+ ja_{k-1,i-1,j} - (j-1)a_{k-1,i-1,j-1} + (3k-i-1)a_{k-1,i-1,j-1}, \end{aligned}$$

which gives (2.8) after simplification. □

Corollary 7. *For $k \geq 0$ and $0 \leq i \leq k$, the coefficients $a_{k,i,j}$ are positive integers for $1 \leq j \leq 2k-i$.*

Proof. This follows from (2.8) by induction on k. Clearly, this is true for $k = 0$ and $k = 1$. Suppose that this is true for some $k \geq 1$. As each term in the right-hand side of (2.8) is nonnegative, we only need to show that at least one term on the right-hand side of (2.8) is strictly positive. Indeed, for $k \geq 2$, the induction hypothesis and (2.8) imply that

- if $j = 1$, then $a_{k,i,1} \geq a_{k-1,i-1,1} > 0$;
- if $2 \leq j \leq 2k-i$, then $a_{k,i,j} \geq (3k-i-j)a_{k-1,i-1,j-1} \geq ka_{k-1,i-1,j-1} > 0$.

These two cases cover all possibilities. $\square$

Theorem 1 follows then from Lemma 5, Proposition 6 and Corollary 7.

Now, define the *Jacobi-Stirling polynomial of the first kind* $g_k(n;z)$ by

$$g_k(n;z) = \mathrm{js}(n, n-k; z). \tag{2.11}$$

Proposition 8. *For $k \geq 1$, we have*

$$g_k(n;z) = f_k(-n;-z). \tag{2.12}$$

If we write $g_k(n;z) = q_{k,0}(n) + q_{k,1}(n)z + \cdots + q_{k,k}(n)z^k$, then

$$\sum_{n\geq 1} q_{k,i}(n)t^n = (-1)^k \frac{\sum_{j=1}^{2k-i} a_{k,i,3k-i+1-j}t^j}{(1-t)^{3k-i+1}}. \tag{2.13}$$

Proof. From (1.2) we deduce

$$\begin{cases} \mathrm{js}(0,0;z) = 1, \qquad \mathrm{js}(n,k;z) = 0, \quad \text{if } k \notin \{1,\dots,n\}, \\ \mathrm{js}(n,k;z) = \mathrm{js}(n-1,k-1;z) - (n-1)(n-1+z)\,\mathrm{js}(n-1,k;z), \quad n,k \geq 1. \end{cases} \tag{2.14}$$

It follows from the above recurrence and (2.11) that

$$g_k(n;z) - g_k(n-1;z) = -(n-1)(n-1+z)g_{k-1}(n-1;z).$$

Comparing with (2.3) we get (2.12), which implies that $q_{k,i}(n) = (-1)^i p_{k,i}(-n)$. Finally (2.13) follows from (2.7). $\square$

3. Jacobi-Stirling posets

We first recall some basic facts about Stanley's theory of P-partitions (see [15] and [16, §4.5]). Let P be a poset, and let ω be a labeling of P, i.e., an injection from P to a totally ordered set (usually a set of integers). A (P,ω)*-partition* (or P-partition if ω is understood) is a function f from P to the positive integers satisfying

(1) if $x <_P y$ then $f(x) \leq f(y)$
(2) if $x <_P y$ and $\omega(x) > \omega(y)$ then $f(x) < f(y)$.

A *linear extension* of a poset P is an extension of P to a total order. We will identify a linear extension of P labeled by ω with the permutation obtained by taking the labels of P in increasing order with respect to the linear extension. For example, the linear extensions of the poset shown in Figure 1 are 2 1 3 and 2 3 1. We write $\mathscr{L}(P)$ for the set of linear extensions of P (which also depend on the labeling ω).

The *order polynomial* $\Omega_P(n)$ of P is the number of (P,ω)-partitions with parts in $[n] = \{1,2,\dots,n\}$. It is known that $\Omega_P(n)$ is a polynomial in n whose degree is the number

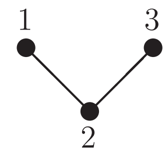

Figure 1. A poset

of elements of P. The following is a fundamental result in the P-partition theory [16, Theorem 4.5.14]:

$$\sum_{n\geq 1}\Omega_P(n)t^n = \frac{\sum_{\pi\in\mathscr{L}(P)} t^{\operatorname{des}\pi+1}}{(1-t)^{k+1}}, \tag{3.1}$$

where k is the number of elements of P and des π is computed according to the natural order of the integers.

For example, the two linear extensions of the poset shown in Figure 1 each have one descent, and the order polynomial for this poset is $2\binom{n+1}{3}$. So equation (3.1) reads

$$\sum_{n\geq 1} 2\binom{n+1}{3} t^n = \frac{2t^2}{(1-t)^4}.$$

By (2.2) the Jacobi-Stirling numbers have the generating function

$$\sum_{n\geq 0}\mathrm{JS}(n,k;z)t^n = \frac{t^k}{(1-(z+1)t)(1-2(z+2)t)\cdots(1-k(z+k)t)}, \tag{3.2}$$

As $f_k(n;z)=\mathrm{JS}(n+k,n;z)$, switching n and k in the last equation yields

$$\sum_{k\geq 0} f_k(n;z)t^k = \frac{1}{(1-(z+1)t)(1-2(z+2)t)\cdots(1-n(z+n)t)}.$$

Identifying the coefficients of t^k gives

$$f_k(n;z) = \sum_{1\leq j_1\leq j_2\leq\cdots\leq j_k\leq n} j_1(z+j_1)\cdot j_2(z+j_2)\cdots j_k(z+j_k). \tag{3.3}$$

For any subset S of $[k]$, we define $\gamma_{S,m}(j)$ by

$$\gamma_{S,m}(j) = \begin{cases} j & \text{if } m\in S,\\ j^2 & \text{if } m\notin S,\end{cases}$$

and define $p_{k,S}(n)$ by

$$p_{k,S}(n) = \sum_{1\leq j_1\leq j_2\leq\cdots\leq j_k\leq n}\gamma_{S,1}(j_1)\gamma_{S,2}(j_2)\cdots\gamma_{S,k}(j_k). \tag{3.4}$$

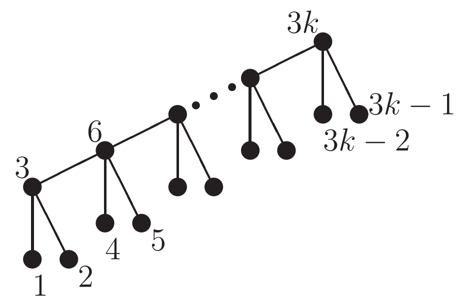

FIGURE 2. The labeled poset R_k.

For example, if $k = 2$ and $S = \{1\}$ then

$$p_{k,S}(n) = \sum_{1\le j_1\le j_2\le n} j_1 j_2^2 = n(n+1)(n+2)(12n^2+9n-1)/120.$$

Definition 3. *Let R_k be the labeled poset in Figure 2. Let S be a subset of $[k]$. The poset $R_{k,S}$ obtained from R_k by removing the points $3m-2$ for $m \in S$ is called a Jacobi-Stirling poset.*

For example, the posets $R_{2,\{1\}}$ and $R_{2,\{2\}}$ are shown in Figure 3.

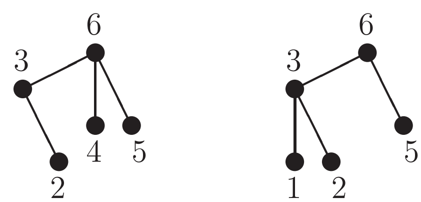

FIGURE 3. The labeled posets $R_{2,\{1\}}$ and $R_{2,\{2\}}$.

Lemma 9. *For any subset $S \subseteq [k]$, let $A_{k,S}(t)$ be the descent polynomial of $\mathscr{L}(R_{k,S})$, i.e., the coefficient of t^j in $A_{k,S}(t)$ is the number of linear extensions of $R_{k,S}$ with $j-1$ descents, then*

$$\sum_{n\ge 0} p_{k,S}(n)t^n = \frac{A_{k,S}(t)}{(1-t)^{3k-|S|+1}}. \tag{3.5}$$

Proof. It is easy to see that $\Omega_{R_{k,S}}(n) = p_{k,S}(n)$ and the result follows from (3.1). $\square$

For $0 \le i \le k$, $R_{k,i}$ is defined as the set of $\binom{k}{i}$ posets

$$R_{k,i} = \{\, R_{k,S} \mid S \subseteq [k] \text{ with cardinality } i \,\}.$$

The posets in $R_{2,1}$ are shown in Figure 3. We define $\mathscr{L}(R_{k,i})$ to be the (disjoint) union of $\mathscr{L}(P)$, over all $P \in R_{k,i}$; i.e.,

$$\mathscr{L}(R_{k,i}) = \bigcup_{\substack{S\subseteq[k]\\|S|=i}} \mathscr{L}(R_{k,S}). \tag{3.6}$$

Now we are ready to give the first interpretation of the coefficients $a_{k,i,j}$ in the polynomial $A_{k,i}(t)$ defined in (2.6).

Theorem 10. *We have*

$$A_{k,i}(t) = \sum_{\substack{S\subseteq[k]\\|S|=i}} A_{k,S}(t). \tag{3.7}$$

In other words, the integer $a_{k,i,j}$ is the number of elements of $\mathscr{L}(R_{k,i})$ with $j-1$ descents.

Proof. Extracting the coefficient of z^i in both sides of (3.3), then applying (1.9) and (3.4), we obtain

$$p_{k,i}(n) = \sum_{\substack{S\subseteq[k]\\|S|=i}} p_{k,S}(n),$$

so that

$$\sum_{n\geq 0} p_{k,i}(n)t^n = \sum_{n\geq 0}\sum_{S} p_{k,S}(n)t^n = \sum_{S}\sum_{n\geq 0} p_{k,S}(n)t^n,$$

where the summations on S are over all subsets of $[k]$ with cardinality i. The result follows then by comparing (2.6) and (3.5). □

It is easy to compute $A_{k,S}(1)$ which is equal to $|\mathscr{L}(R_{k,S})|$ and is also $(3k-i)!$ times the leading coefficient of $p_{k,S}(n)$.

Proposition 11. *Let $S \subseteq [k]$, $|S| = i$ and let $l_j(S) = |\{\, s \in S \mid s \leq j \,\}|$ for $1 \leq j \leq k$. We have*

$$A_{k,S}(1) = \frac{(3k-i)!}{\prod_{j=1}^{k}(3j - l_j(S))}. \tag{3.8}$$

Proof. We construct a permutation in $\mathscr{L}(R_{k,S})$ by reading the elements of $R_{k,S}$ in increasing order of their labels and inserting each one into the permutation already constructed from the earlier elements. Each element of $R_{k,S}$ will have two natural numbers associated to it: the reading number and the insertion-position number. It is clear that the insertion-position number of $3j$ must be equal to its reading number, which is $3j - l_j(S)$, since it must be inserted to the right of all the previously inserted elements (those with labels less than $3j$). On the other hand, an element not divisible by 3 may be inserted anywhere, so its number of possible insertion positions is equal to its reading number. So the number of possible linear extensions of $R_{k,S}$ is equal to the product of the reading numbers of all elements with labels not divisible by 3. Since the product of all the reading

numbers is $(3j-i)!$, we obtain the result by dividing this number by the product of the reading numbers of the elements with labels $3, 6, \dots, 3k$. $\square$

From (3.8) we can derive the formula for $A_{k,i}(1)$, which is equivalent to Proposition 3.

Proposition 12. *We have*

$$|\mathscr{L}(R_{k,i})| = A_{k,i}(1) = \frac{(3k-i)!}{3^{k-i}2^i\, i!\,(k-i)!}.$$

Proof. By Proposition 11 it is sufficient to prove the identity

$$\sum_{1\le s_1<\cdots<s_i\le k} \frac{(3k-i)!}{\prod_{j=1}^{k}(3j-l_j(S))} = \frac{(3k-i)!}{3^{k-i}2^i\, i!\,(k-i)!}, \tag{3.9}$$

where $S=\{s_1,\dots,s_i\}$ and $l_j(S) = |\{\, s\in S \,:\, s\le j\,\}|$.

The identity is obvious if $S=\varnothing$, i.e., $i=0$. When $i=1$, it is easy to see that (3.9) is equivalent to the $a=2/3$ case of the indefinite summation

$$\sum_{s=0}^{k-1} \frac{(a)_s}{s!} = \frac{(a+1)_{k-1}}{(k-1)!}, \tag{3.10}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ and $(a)_0=1$. Since the left-hand side of (3.9) can be written as

$$\sum_{s_i=i}^{k} \frac{(3k-i)!}{\prod_{j=s_i}^{k}(3j-i)} \sum_{1\le s_1<\cdots<s_{i-1}\le s_i-1} \frac{1}{\prod_{j=1}^{s_i-1}(3j-l_j(S))}, \tag{3.11}$$

we derive (3.9) from the induction hypothesis and (3.10). $\square$

Remark 1. *Alternatively, we may prove the formula for $A_{k,i}(1)$ as follows:*

$$\begin{aligned} A_{k,i}(1) &= \sum_{\substack{S\subseteq[k]\\ |S|=i}} A_{k,S}(1) \\ &= \sum_{\substack{S\subseteq[k]\\ |S|=i,k\in S}} A_{k,S}(1) + \sum_{\substack{S\subseteq[k]\\ |S|=i,k\notin S}} A_{k,S}(1) \\ &= (3k-i-1)A_{k-1,i-1}(1) + (3k-i-1)(3k-i-2)A_{k-1,i}(1), \end{aligned}$$

from which we easily deduce that $A_{k,i}(1) = (3k-i)!/3^{k-i}2^i\, i!\,(k-i)!$.

Since both of the above proofs of Proposition 12 use mathematical induction, it is desirable to have a more conceptual proof. Here we give such a proof based on the fact that Proposition 12 is equivalent to

$$|\mathscr{L}(R_{k,i})| = 2^{k-i}\cdot \frac{(3k-i)!}{2!^i\, i!\, 3!^{k-i}(k-i)!}. \tag{3.12}$$

A combinatorial proof of Proposition 12. We show that $|\mathscr{L}(R_{k,i})|$ is equal to 2^{k-i} times the number of partitions of $[3k-i]$ with $k-i$ blocks of size 3 and i blocks of size 2.

Let S be an i-element subset of $[k]$ and let π be an element $\mathscr{L}(R_{k,S})$, viewed as a bijection from $[3k-i]$ to $R_{k,S}$. Let $\sigma = \pi^{-1}$. Then σ is a natural labeling of $R_{k,S}$, i.e., an order-preserving bijection from the poset $R_{k,S}$ to $[3k-i]$, and conversely, every natural labeling of $R_{k,S}$ is the inverse of an element of $\mathscr{L}(R_{k,S})$.

We will describe a map from the set of natural labelings of elements of $R_{k,i}$ to the set of partitions of $[3k-i]$ with $k-i$ blocks of size 3 and i blocks of size 2, for which each such partition is the image of 2^{k-i} natural labelings. Given a natural labeling σ of $R_{k,S}$, the blocks of the corresponding partition are the sets $\{\sigma(3m-2), \sigma(3m-1), \sigma(3m)\}$ for $m \notin S$ and the sets $\{\sigma(3m-1), \sigma(3m)\}$ for $m \in S$. We note that since σ is a natural labeling, $\sigma(3m)$ is always the largest element of its block and $\sigma(3) < \sigma(6) < \cdots < \sigma(3m)$.

Now let P be a partition of $[3k-i]$ with $k-i$ blocks of size 3 and i blocks of size 2. We shall describe all natural labelings σ of posets $R_{k,S}$ that correspond to P under the map just defined. First, we list the blocks of P as $B_1, B_2, \ldots, B_k$ in increasing order of their largest elements. Then $\sigma(3m)$ must be the largest element of B_m. If B_m has two elements, then the smaller element must be $\sigma(3m-1)$, and m must be an element of S. If B_m has three elements then $m \notin S$, and $\sigma(3m-2)$ and $\sigma(3m-1)$ are the two smaller elements of B_m, but in either order. Thus S is uniquely determined by P, and there are exactly 2^{k-i} natural labelings of $R_{k,S}$ in the preimage of P. So $|\mathscr{L}(R_{k,i})|$ is 2^i times the number of partitions of $[3k-i]$ with $k-i$ blocks of size 3 and i blocks of size 2, and is therefore equal to the right-hand side of (3.12). $\square$

4. Two proofs of Theorem 2

We shall give two proofs of Theorem 2. We first derive Theorem 2 from Theorem 10 by constructing a bijection from the linear extensions of Jacobi-Stirling posets to permutations. The second proof consists of verifying that the cardinality of Jacobi-Stirling permutations in $\mathcal{JSP}_{k,i}$ with $j-1$ descents satisfies the recurrence relation (2.8). Given a word $w = w_1 w_2 \ldots w_m$ of m letters, we define the jth slot of w by the pair (w_j, w_{j+1}) for $j = 0, \ldots, m$. By convention $w_0 = w_{m+1} = 0$. A slot (w_j, w_{j+1}) is called a descent (resp. non-descent) slot if $w_j > w_{j+1}$ (resp. $w_j \le w_{j+1}$).

4.1. **First proof of Theorem 2.** For any subset $S = \{s_1, \ldots, s_i\}$ of $[k]$ we define $\bar{S} = \{\bar{s}_1, \ldots, \bar{s}_i\}$, which is a subset of $[\bar{k}]$. Recall that $\mathcal{JSP}_{k,\bar{S}}$ is the set of Jacobi-Stirling permutations of $M_{k,\bar{S}}$. We construct a bijection $\phi : \mathscr{L}(R_{k,S}) \to \mathcal{JSP}_{k,\bar{S}}$ such that $\operatorname{des} \phi(\pi) = \operatorname{des} \pi$ for any $\pi \in \mathscr{L}(R_{k,S})$.

If $k = 1$, then $\mathscr{L}(R_{1,0}) = \{123,\ 213\}$ and $\mathscr{L}(R_{1,1}) = \{23\}$. We define ϕ by

$$\phi(123) = \bar{1}11,\ \phi(213) = 11\bar{1},\ \phi(23) = 11.$$

Suppose that $k \ge 2$ and $\phi : \mathscr{L}(R_{k-1,S}) \to \mathcal{JSP}_{k-1,\bar{S}}$ is defined for any $S \subseteq [k-1]$. If $\pi \in \mathscr{L}(R_{k,S})$ with $S \subseteq [k]$, we consider the following two cases:

(i) $k \notin S$, denote by π' the word obtained by deleting $3k$ and $3k-1$ from π, and π'' the word obtained by further deleting $3k-2$ from π'. As $\pi'' \in \mathscr{L}(R_{k-1,S})$, by induction hypothesis, the permutation $\phi(\pi'') \in \mathcal{JSP}_{k-1,\bar{S}}$ is well defined. Now,
  a) if $3k-2$ is in the rth descent (or nondescent) slot of π'', then we insert $\bar{k}$ in the rth descent (or nondescent) slot of $\phi(\pi'')$ and obtain a word $\phi_1(\pi'')$;
  b) if $3k-1$ is in the sth descent (or nondescent) slot of π', we define $\phi(\pi)$ by inserting kk in the sth descent (or nondescent) slot of $\phi_1(\pi'')$.

(ii) $k \in S$, denote by π' the word obtained from π by deleting $3k$ and $3k-1$. As $\pi' \in \mathscr{L}(R_{k-1,i-1})$, the permutation $\phi(\pi') \in \mathcal{JSP}_{k-1,\bar{S}}$ is well defined. If $3k-1$ is in the rth descent (or nondescent) slot of π', we define $\phi(\pi)$ by inserting kk in the rth descent (or nondescent) slot of $\phi(\pi')$.

Clearly this mapping is a bijection and preserves the number of descents. For example, if $k=3$ and $S=\{2\}$, then $\phi(25137869) = 112\bar{3}233\bar{1}$. This can be seen by applying the mapping ϕ as follows:

$$213 \to 25136 \to 251376 \to 25137869,$$
$$11\bar{1} \to 1122\bar{1} \to 112\bar{3}2\bar{1} \to 112\bar{3}233\bar{1}.$$

Clearly we have des$(2\mathbf{5}137\mathbf{8}69) = 2$ and des$(112\bar{\mathbf{3}}23\mathbf{3}\bar{1}) = 2$.

4.2. **Second proof of Theorem 2.** Let $\mathcal{JSP}_{k,i,j}$ be the set of Jacobi-Stirling permutations in $\mathcal{JSP}_{k,i}$ with $j-1$ descents. Let $a'_{0,0,0} = 1$ and $a'_{k,i,j}$ be the cardinality of $\mathcal{JSP}_{k,i,j}$ for $k,i,j \geq 0$. By definition, $a'_{k,i,j} = 0$ if any of the indices $k,i,j < 0$ or $j \notin \{1,\dots,2k-i\}$. We show that $a'_{k,i,j}$'s satisfy the same recurrence (2.8) and initial conditions as $a_{k,i,j}$'s.

Any Jacobi-Stirling permutation of $\mathcal{JSP}_{k,i,j}$ can be obtained from one of the following five cases:

(i) Choose a Jacobi-Stirling permutation in $\mathcal{JSP}_{k-1,i,j}$, insert $\bar{k}$ and then kk in one of the descent slots (an extra descent at the end of the permutation). Clearly, there are $a'_{k-1,i,j}$ ways to choose the initial permutation, j ways to insert $\bar{k}$, and j ways to insert kk.

(ii) Choose a Jacobi-Stirling permutation of $\mathcal{JSP}_{k-1,i,j-1}$,
  1) insert $\bar{k}$ in a descent slot and then kk in a non-descent slot. In this case, there are $a'_{k-1,i,j-1}$ ways to choose the initial permutation, $j-1$ ways to insert $\bar{k}$, and $3k-i-j-1$ ways to insert kk.
  2) insert $\bar{k}$ in a non-descent slot and then kk in a descent slot. In this case, there are $a'_{k-1,i,j-1}$ ways to choose the initial permutation, $3k-i-j-1$ ways to insert $\bar{k}$, and j ways to insert kk.

(iii) Choose a Jacobi-Stirling permutation in $\mathcal{JSP}_{k-1,i,j-2}$, insert $\bar{k}$ and then kk in one of the non-descent slots. In this case, there are $a'_{k-1,i,j-2}$ ways to choose the initial permutation, $3k-i-j$ ways to insert $\bar{k}$, and $3k-i-j$ ways to insert kk.

(iv) Choose a Jacobi-Stirling permutation in $\mathcal{JSP}_{k-1,i-1,j}$ and insert kk in one of the descent slots. There are $a'_{k-1,i-1,j}$ ways to choose the initial permutation, and j ways to insert kk.

(v) Choose a Jacobi-Stirling permutation in $\mathcal{JSP}_{k-1,i-1,j-1}$ and insert kk in one of the non-descent slots. There are $a'_{k-1,i-1,j-1}$ ways to choose the initial permutation, and $3k-i-j$ ways to insert kk.

Summarizing all the above five cases, we obtain

$$\begin{aligned} a'_{k,i,j} &= j^2 a'_{k-1,i,j} + [2(j-1)(3k-i-j-1) + (3k-i-2)]a'_{k-1,i,j-1} \\ &\quad + (3k-i-j)^2 a'_{k-1,i,j-2} + j a'_{k-1,i-1,j} + (3k-i-j)a'_{k-1,i-1,j-1}. \end{aligned}$$

Therefore, the numbers $a'_{k,i,j}$ satisfy the same recurrence and initial conditions as the $a_{k,i,j}$, so they are equal.

5. Legendre-Stirling posets

Let P_k be the poset shown in Figure 4, called the *Legendre-Stirling poset*. The order

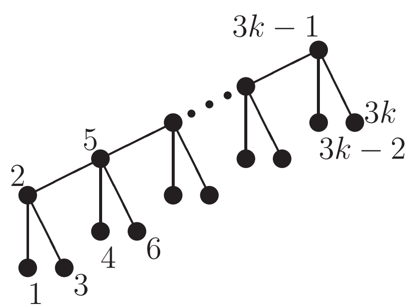

Figure 4. The Legendre-Stirling poset P_k.

polynomial of P_k is given by

$$\begin{aligned} \Omega_{P_k}(n) &= \sum_{2 \le f(2) \le \cdots \le f(3k-1) \le n} \prod_{i=1}^{k} f(3i-1)(f(3i-1)-1) \\ &= [x^k] \frac{1}{(1-2x)(1-6x)\dots(1-(n-1)nx)}, \end{aligned}$$

which is equal to $\mathrm{JS}(n-1+k, n-1; 1)$ by (3.2), and by (1.3) this is equal to $\mathrm{LS}(n-1+k, n-1)$. By (3.1), we obtain

$$\sum_{n\ge 0} \mathrm{LS}(n+k,n)t^n = \frac{\sum_{\pi\in\mathscr{L}(P_k)} t^{\operatorname{des}\pi}}{(1-t)^{3k+1}}. \tag{5.1}$$

In other words, we have the following theorem.

Theorem 13. *Let $b_{k,j}$ be the number of linear extensions of Legendre-Stirling posets P_k with exactly j descents. Then*

$$\sum_{n\geq 0} \mathrm{LS}(n+k,n)t^n = \frac{\sum_{j=1}^{2k-1} b_{k,j}t^j}{(1-t)^{3k+1}}. \tag{5.2}$$

We now apply the above theorem to deduce a result of Egge [5, Theorem 4.6].

Definition 4. *A Legendre-Stirling permutation of M_k is a Jacobi-Stirling permutation of M_k with respect to the order: $\bar{1} = 1 < \bar{2} = 2 < \cdots < \bar{k} = k$.*

Here $\bar{1} = 1$ means that neither $1\bar{1}$ nor $\bar{1}1$ counts as a descent. Thus, the Legendre-Stirling permutation $122\bar{2}1\bar{1}$ has one descent at position 4, while as a Jacobi-Stirling permutation, it has three descents, at positions 3, 4 and 5.

Theorem 14 (Egge)**.** *The coefficient $b_{k,j}$ equals the number of Legendre-Stirling permutations of M_k with exactly $j-1$ descents.*

First proof. Let $\mathcal{LSP}_k$ be the set of Legendre-Stirling permutations of M_k. By Theorem 13, it suffices to construct a bijection $\psi : \mathcal{LSP}_k \to \mathscr{L}(P_k)$ such that $\operatorname{des}\psi(\pi) - 1 = \operatorname{des}\pi$ for any $\pi \in \mathcal{LSP}_k$. If $k = 1$, then $\mathcal{LSP}_1 = \{11\bar{1},\ \bar{1}11\}$ and $\mathscr{L}(P_1)$=$\{132,\ 312\}$. We define ψ by

$$\psi(11\bar{1}) = 132, \quad \psi(\bar{1}11) = 312.$$

Clearly $\operatorname{des} 132 - 1 = \operatorname{des} 11\bar{1} = 0$ and $\operatorname{des} 312 - 1 = \operatorname{des} \bar{1}11 = 0$. Suppose that the bijection $\psi : \mathcal{LSP}_{k-1} \to \mathscr{L}(P_{k-1})$ is constructed for some $k \geq 2$. Given $\pi \in \mathcal{LSP}_k$, we denote by π' the word obtained by deleting $\bar{k}$ from π, and by π'' the word obtained by further deleting kk from π'. We put $3k-1$ at the end of $\psi(\pi'')$ and obtain a word $\psi_1(\pi'')$. In the following two steps, the slot after $3k-1$ is excluded, because we cannot insert $3k$ and $3k-2$ to the right of $3k-1$.

a) if $\bar{k}$ is in the rth descent (or nondescent) slot of π'', then we insert $3k$ in the rth descent (or nondescent) slot of $\psi_1(\pi'')$ and obtain a word $\psi_2(\pi'')$;
b) if kk is in the sth descent slot or in the non-descent slot before $\bar{k}$ (in the jth non-descent slot other than the non-descent slot before $\bar{k}$) of π', we define $\psi(\pi)$ by inserting $3k-2$ in the sth descent slot or in the non-descent slot before $3k$ (in the jth non-descent slot other than the non-descent slot before $3k$) of $\psi_2(\pi'')$.

For example, we can compute $\psi(\bar{2}122333\bar{3}1\bar{1}) = 614793258$ by the following procedure:

$$11\bar{1} \to \bar{2}11\bar{1} \to \bar{2}12211\bar{1} \to \bar{2}122\bar{3}1\bar{1} \to \bar{2}122333\bar{3}1\bar{1}$$

$$132 \to 61325 \to 614325 \to 61493258 \to 614793258.$$

This construction can be easily reversed and the number of descents is preserved. $\square$

Second proof. By (1.8), (1.9), and (1.10), we have

$$\sum_{n=0}^{\infty} \mathrm{JS}(n+k,n;z)t^n = \sum_{i=0}^{k} z^i \frac{\sum_{j=1}^{2k-i} a_{k,i,j}t^j}{(1-t)^{3k-i+1}}.$$

Setting $z=1$ and using (1.3) gives

$$\sum_{n=0}^{\infty} \mathrm{LS}(n+k,n)t^n = \sum_{i=0}^{k}(1-t)^i \frac{\sum_{j=1}^{2k-i} a_{k,i,j}t^j}{(1-t)^{3k+1}}.$$

Multiplying both sides by $(1-t)^{3k+1}$ and applying (5.2) gives

$$\sum_{j=1}^{2k-1} b_{k,j}t^j = \sum_{i=0}^{k}(1-t)^i \sum_{j=1}^{2k-i} a_{k,i,j}t^j,$$

so

$$\sum_{i=0}^{k}\sum_{l=0}^{i}(-1)^l \binom{i}{l} a_{k,i,j-l} = b_{k,j}. \tag{5.3}$$

For any $S \subseteq [\bar{k}]$, let $\mathcal{JSP}_{k,S,j}$ be the set of all Jacobi-Stirling permutations of $M_{k,S}$ with $j-1$ descents. Let $B_{k,j} = \bigcup_{S\subseteq[\bar{k}]} \mathcal{JSP}_{k,S,j}$ be the set of Jacobi-Stirling permutations with $j-1$ descents. We show that the left-hand side of (5.3) is the number N_0 of permutations in $B_{k,j}$ with no pattern $u\bar{u}$.

For any $T \subseteq [\bar{k}]$, let $B_{k,j}(T,\geq)$ be the set of permutations in $B_{k,j}$ containing all the patterns $u\bar{u}$ for $\bar{u} \in T$. By the principle of inclusion-exclusion [16, Chapter 2],

$$N_0 = \sum_{T\subseteq[\bar{k}]} (-1)^{|T|} |B_{k,j}(T,\geq)|. \tag{5.4}$$

Now, for any subsets $T, S \subseteq [\bar{k}]$ such that $T \subseteq [\bar{k}] \setminus S$, define the mapping

$$\varphi : \mathcal{JSP}_{k,S,j} \cap B_{k,j}(T,\geq) \to \mathcal{JSP}_{k,S\cup T,j-|T|}$$

by deleting the $\bar{u}$ in every pattern $u\bar{u}$ of $\pi \in \mathcal{JSP}_{k,S,j} \cap B_{k,j}(T,\geq)$. Clearly, this is a bijection. Hence, we can rewrite (5.4) as

$$\begin{aligned} N_0 &= \sum_{T\subseteq[\bar{k}]} (-1)^{|T|} \sum_{\substack{S,T\subseteq[\bar{k}]\\ T\cap S=\emptyset}} |\mathcal{JSP}_{k,S\cup T,j-|T|}| \\ &= \sum_{T\subseteq[\bar{k}]} (-1)^{|T|} \sum_{\substack{S\subseteq[\bar{k}]\\ T\subseteq S}} |\mathcal{JSP}_{k,S,j-|T|}|. \end{aligned}$$

For any subset S of $[\bar{k}]$ with $|S|=i$, and any l with $0 \leq l \leq i$, there are $\binom{i}{l}$ subsets T of S such that $|T|=l$, and, by definition,

$$\sum_{\substack{S\subseteq[\bar{k}]\\ |S|=i}} |\mathcal{JSP}_{k,S,j-|T|}| = a_{k,i,j-l}.$$

This proves that N_0 is equal to the left-hand side of (5.3).

Let $\mathcal{LSP}_{k,j}$ be the set of all Legendre-Stirling permutations of M_k with $j-1$ descents. It is easy to identify a permutation $\pi \in B_{k,j}$ with no pattern $u\bar{u}$ with a Legendre-Stirling

permutation $\pi' \in \mathcal{LSP}_{k,j}$ by inserting each missing $\bar{u}$ just to the right of the second u. This completes the proof. $\square$

Finally, the numerical experiments suggest the following conjecture, which has been verified for $0 \le i \le k \le 9$.

Conjecture 15. *For $0 \le i \le k$, the polynomial $A_{k,i}(t)$ has only real roots.*

Note that by a classical result [4, p. 141], the above conjecture would imply that the sequence $a_{k,i,1}, \dots, a_{k,i,2k-i}$ is unimodal. Let G_k be the multiset $\{1^{m_1}, 2^{m_2}, \dots, k^{m_k}\}$ with $m_i \in \mathbb{N}$. A permutations π of G_k is a *generalized Stirling permutation* (see [3, 12]) if whenever $u < v < w$ and $\pi(u) = \pi(w)$, we have $\pi(v) > \pi(u)$. For any $S \subseteq [\bar{k}]$, the set of generalized Stirling permutations of $M_k \setminus S$ is equal to $\mathcal{JSP}_{k,S}$. By Lemma 9 and Theorem 2, the descent polynomial of $\mathcal{JSP}_{k,S}$ is $A_{k,S}(t)$. It follows from a result of Brenti [3, Theorem 6.6.3] that $A_{k,S}(t)$ has only real roots. By (3.7), this implies, in particular, that the above conjecture is true for $i = 0$ and $i = k$.

One can also use the methods of Haglund and Visontai [11] to show that $A_{k,S}(t)$ has only real roots, though it is not apparent how to use these methods to show that $A_{k,i}(t)$ has only real roots.

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